Unit 5 – Analytical Applications of Differentiation

Properties of First Derivatives:

**Derivative** is a rate of change: it finds the change in $y$ over the change in $x$, $\frac{dy}{dx}$, which is slope.

*1st derivative* $\Rightarrow$ max. and min., increasing and decreasing, slope of the tangent line to the curve, and velocity.

*2nd derivative* $\Rightarrow$ inflection points, concavity, and acceleration.

**Properties of First Derivative:**

**Increasing:** slopes of tangent lines are _______________ $f'(x) > ____$

**Decreasing:** slopes of tangent lines are _______________ $f'(x) < ____$

**Maximum Point:** Set $f'(x) = ____$ and it is where the slopes turn from ______________ to ____________

**Minimum Point:** Set $f'(x) = ____$ and it is where the slopes turn from ______________ to ____________

Properties of Second Derivative:

**Concave Up:** slopes of tangent lines are _______________ $f''(x) > ____$

**Concave Down:** slopes of tangent lines are _______________ $f''(x) < ____$

**Inflection Points:** Set $f''(x) = ____$ and it is where the points on the graph switch ______________

EX#1: (a) Find the maximum point, the minimum point, the intervals of increasing, and the intervals of decreasing for the following function:

$$y = 2x^3 - 3x^2 - 36x + 2$$

(b) Find the inflection point, and the intervals of concavity for the function in EX#1.

Optimization Problems:

1) Draw and label a picture.
2) Write equations that fit the scenario.
3) Combine equations into one equation.
4) Take the derivative and set it equal to 0.
5) Solve for the variable.
Mean Value Theorem of Derivatives:

The slope of the tangent at value c _____________ the slope of the secant through a and b.

EX#1: For what value c, such that 0 ≤ c ≤ 3, is the instantaneous rate of change for \( f(x) = x^2 - 2x \) equal to the average rate of change over the interval [0, 3]?
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Sample AP Problems:

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5. If \( g \) is the function given by \( g(x) = \frac{1}{3}x^3 + \frac{3}{2}x^2 - 70x + 5 \), on which of the following intervals is \( g \) decreasing?
   (A) \((-\infty, -10)\) and \((7, \infty)\)
   (B) \((-\infty, -7)\) and \((10, \infty)\)
   (C) \((-\infty, 10)\)
   (D) \((-10, 7)\)
   (E) \((-7, 10)\)

8. A particle moves along a straight line. The graph of the particle’s velocity \( v(t) \) at time \( t \) is shown above for \( 0 \leq t \leq m \), where \( j, k, l, \) and \( m \) are constants. The graph intersects the horizontal axis at \( t = 0, t = k, \) and \( t = m \) and has horizontal tangents at \( t = j \) and \( t = l \). For what values of \( t \) is the speed of the particle decreasing?
   (A) \( j \leq t \leq l \)
   (B) \( k \leq t \leq m \)
   (C) \( j \leq t \leq k \) and \( l \leq t \leq m \)
   (D) \( 0 \leq t \leq j \) and \( k \leq t \leq l \)
   (E) \( 0 \leq t \leq j \) and \( l \leq t \leq m \)

13. Let \( f \) be a differentiable function such that \( f(0) = -5 \) and \( f'(x) \leq 3 \) for all \( x \). Of the following, which is not a possible value for \( f(2) \)?
   (A) \(-10\)  (B) \(-5\)  (C) \(0\)  (D) \(1\)  (E) \(2\)

24. The function \( g \) is given by \( g(x) = 4x^3 + 3x^2 - 6x + 1 \). What is the absolute minimum value of \( g \) on the closed interval \([-2, 1]\)?
   (A) \(-7\)  (B) \(-\frac{3}{4}\)  (C) \(0\)  (D) \(2\)  (E) \(6\)
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28. The function \( f \) is defined by \( f(x) = \sin x + \cos x \) for \( 0 \leq x \leq 2\pi \). What is the x-coordinate of the point of inflection where the graph of \( f \) changes from concave down to concave up?
   
   (A) \( \frac{\pi}{4} \)  
   (B) \( \frac{3\pi}{4} \)  
   (C) \( \frac{5\pi}{4} \)  
   (D) \( \frac{7\pi}{4} \)  
   (E) \( \frac{9\pi}{4} \)

82. The derivative of the function \( f \) is given by \( f''(x) = x^3 - 4\sin(x^2) + 1 \). On the interval \((-2.5, 2.5)\), at which of the following values of \( x \) does \( f \) have a relative maximum?
   
   (A) \(-1.970\) and 0
   (B) \(-1.467\) and 1.075
   (C) \(-0.475, 0.542, \) and 1.396
   (D) \(-0.475\) and 1.396 only
   (E) 0.542 only

86. If \( f'(x) > 0 \) for all \( x \) and \( f''(x) < 0 \) for all \( x \), which of the following could be a table of values for \( f \)?

   (A) \[
   \begin{array}{c|c}
   x & f(x) \\
   \hline
   -1 & 4 \\
   0 & 3 \\
   1 & 1 \\
   \end{array}
   \]
   (B) \[
   \begin{array}{c|c}
   x & f(x) \\
   \hline
   -1 & 4 \\
   0 & 4 \\
   1 & 4 \\
   \end{array}
   \]
   (C) \[
   \begin{array}{c|c}
   x & f(x) \\
   \hline
   -1 & 4 \\
   0 & 5 \\
   1 & 6 \\
   \end{array}
   \]
   (D) \[
   \begin{array}{c|c}
   x & f(x) \\
   \hline
   -1 & 4 \\
   0 & 5 \\
   1 & 7 \\
   \end{array}
   \]
   (E) \[
   \begin{array}{c|c}
   x & f(x) \\
   \hline
   -1 & 4 \\
   0 & 6 \\
   1 & 7 \\
   \end{array}
   \]

87. Let \( f \) be the function with first derivative given by \( f'(x) = (3 - 2x - x^2) \sin(2x - 3) \). How many relative extrema does \( f \) have on the open interval \(-4 < x < 2\)?

   (A) Two    (B) Three    (C) Four    (D) Five    (E) Six

88. The graph of a twice-differentiable function \( f \) is shown in the figure above. Which of the following is true?

   (A) \( f''(-1) < f'(-1) < f'(0) \)
   (B) \( f''(-1) < f'(0) < f'(-1) \)
   (C) \( f'(0) < f''(-1) < f'(-1) \)
   (D) \( f'(-1) < f''(-1) < f'(0) \)
   (E) \( f'(-1) < f'(0) < f'(-1) \)
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92. The function \( f \) is defined for all \( x \) in the closed interval \([a, b]\). If \( f \) does not attain a maximum value on \([a, b]\), which of the following must be true?

(A) \( f \) is not continuous on \([a, b]\).

(B) \( f \) is not bounded on \([a, b]\).

(C) \( f \) does not attain a minimum value on \([a, b]\).

(D) The graph of \( f \) has a vertical asymptote in the interval \([a, b]\).

(E) The equation \( f'(x) = 0 \) does not have a solution in the interval \([a, b]\).

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9. The function \( f \) has a first derivative given by \( f'(x) = x(x - 3)^2(x + 1) \). At what values of \( x \) does \( f \) have a relative maximum?

(A) \(-1\) only

(B) \(0\) only

(C) \(-1\) and \(0\) only

(D) \(-1\) and \(3\) only

(E) \(-1\), \(0\), and \(3\)

15. The function \( y = g(x) \) is differentiable and increasing for all real numbers. On what intervals is the function \( y = g(x^3 - 6x^2) \) increasing?

(A) \((-\infty, 0]\) and \([4, \infty)\) only

(B) \([0, 4]\) only

(C) \([2, \infty)\) only

(D) \([6, \infty)\) only

(E) \((-\infty, \infty)\)

19. For what values of \( x \) does the graph of \( y = 3x^5 + 10x^4 \) have a point of inflection?

(A) \(x = -\frac{8}{3}\) only

(B) \(x = -2\) only

(C) \(x = 0\) only

(D) \(x = 0\) and \(x = -\frac{8}{3}\)

(E) \(x = 0\) and \(x = -2\)

22. Let \( f \) be the function defined by \( f(x) = 2x^3 - 3x^2 - 12x + 18 \). On which of the following intervals is the graph of \( f \) both decreasing and concave up?

(A) \((-\infty, -1)\)

(B) \((-1, \frac{1}{2})\)

(C) \((-1, 2)\)

(D) \(\left[\frac{1}{2}, 2\right]\)

(E) \((2, \infty)\)
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78. The function \( f \) is differentiable and increasing for all real numbers \( x \), and the graph of \( f \) has exactly one point of inflection. Of the following, which could be the graph of \( f' \), the derivative of \( f \)?

(A) \[
\begin{array}{c}
\text{y} \\
\hline
O \\
\text{x}
\end{array}
\]

(B) \[
\begin{array}{c}
\text{y} \\
\hline
O \\
\text{x}
\end{array}
\]

(C) \[
\begin{array}{c}
\text{y} \\
\hline
O \\
\text{x}
\end{array}
\]

(D) \[
\begin{array}{c}
\text{y} \\
\hline
O \\
\text{x}
\end{array}
\]

(E) \[
\begin{array}{c}
\text{y} \\
\hline
O \\
\text{x}
\end{array}
\]

80. The table above gives selected values of a function \( f \). The function is twice differentiable with \( f''(x) > 0 \). Which of the following could be the value of \( f'(3) \)?

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.4</td>
</tr>
<tr>
<td>3</td>
<td>3.6</td>
</tr>
<tr>
<td>5</td>
<td>5.4</td>
</tr>
</tbody>
</table>

(A) 0.6  (B) 0.7  (C) 0.9  (D) 1.2  (E) 1.5
82. The figure above shows the graph of $f'$, the derivative of function $f$, for $-6 < x < 8$. Of the following, which best describes the graph of $f$ on the same interval?

(A) 1 relative minimum, 1 relative maximum, and 3 points of inflection
(B) 1 relative minimum, 1 relative maximum, and 4 points of inflection
(C) 2 relative minima, 1 relative maximum, and 2 points of inflection
(D) 2 relative minima, 1 relative maximum, and 4 points of inflection
(E) 2 relative minima, 2 relative maxima, and 3 points of inflection

91. Let $F$ be a function defined for all real numbers $x$ such that $F'(x) > 0$ and $F''(x) > 0$. Which of the following could be a table of values for $F$?

(A) $\begin{array}{c|c} x & F(x) \\ \hline 1 & -3 \\ 2 & -4 \\ 3 & -6 \\ 4 & -9 \end{array}$

(B) $\begin{array}{c|c} x & F(x) \\ \hline 1 & -3 \\ 2 & -1 \\ 3 & 3 \\ 4 & 19 \end{array}$

(C) $\begin{array}{c|c} x & F(x) \\ \hline 1 & -3 \\ 2 & 0 \\ 3 & 3 \\ 4 & 6 \end{array}$

(D) $\begin{array}{c|c} x & F(x) \\ \hline 1 & -3 \\ 2 & 5 \\ 3 & 11 \\ 4 & 13 \end{array}$

(E) $\begin{array}{c|c} x & F(x) \\ \hline 1 & -3 \\ 2 & -4 \\ 3 & -3 \\ 4 & 0 \end{array}$